

Quantum Mechanics as a Classical Theory II: Relativistic Theory

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Abstract

In this article, the axioms presented in the first one are reformulated according to the special theory of relativity. Using these axioms, quantum mechanic's relativistic equations are obtained in the presence of electromagnetic fields for both the density function and the probability amplitude. It is shown that, within the present theory's scope, Dirac's second order equation should be considered the fundamental one in spite of the first order equation. A relativistic expression is obtained for the statistical potential. Axioms are again altered and made compatible with the general theory of relativity. These postulates, together with the idea of the statistical potential, allow us to obtain a general relativistic quantum theory for *ensembles* composed of single particle systems.

1 Introduction

The first paper of this series demonstrated how, accepting a few axioms, quantum mechanics can be derived from newtonian mechanics. Amongst these axioms was the validity of the Wigner-Moyal Infinitesimal Transformation. One variation of this transformation has already been amply studied[1]-[11] and the conclusion was that quantum mechanics cannot be derived from these transformations, because the density function obtained is not positive definite[12]-[15]. Various efforts to adapt a non-classical phase space to quantum mechanics followed these frustrated attempts. These attempts basically assume that quantum phase space cannot contain isolated points - which would not make sense because of the uncertainty relations - but instead regions, with dimensions related to the quantum of action. These spaces are called stochastic phase spaces[16]-[23].

We emphasize once more that the transformation here presented is distinct from that presented in the literature cited above. In the form here presented, it is only a mathematical instrument to obtain probability densities in configuration

space, using the limiting process already described, from the joint probability density function defined in phase space. In this manner it does not present the positivity problem, as was demonstrated in the previous paper. Strictly speaking, this transformation would not even need to constitute one of the theory's axioms, and we only treat it as such to emphasize the differences between it and the one generally used. It must be stressed that we are working with classical phase space in this series of papers and that our axioms are of a purely classical character. Moreover, we demonstrated in the first paper, that the uncertainty relations are a consequence of the adopted formalism and not a fundamental property of nature; so there is no reason to limit our system's description to stochastic phase space.

In this second paper, we will show that both Klein-Gordon's and Dirac's relativistic equations for the density function and for the probability amplitude can be derived from small alterations in our axioms, made to adapt quantum theory to the special theory of relativity. We will also include the electromagnetic field in our considerations in order to obtain Dirac's equation. Contrary to what is usually accepted, the fundamental character of Dirac's second order equation will be established, instead of his first order equation.

Once again, changing the postulates in order to adapt them to the general theory of relativity and using the statistical potential concept, we will demonstrate that it is possible to obtain a system of general-relativistic-quantum equations, which takes into account the gravitational field's effects, for an *ensemble* of one particle systems.

In the second section, we will develop the special relativistic formalism, obtaining Klein-Gordon's and Dirac's equations for both the density function and the probability amplitude. We will obtain an expression for the relativistic statistical potential to be used in the general relativistic treatment.

In the third section, we will obtain the system of general relativistic quantum equations which includes, in the quantum mechanical treatment of one particle system *ensembles*, the effects of the gravitational field.

Our conclusions will be developed in the final section.

In the appendix, we will show the relation between the density function calculated in four dimensional space, which is a τ -constant of motion, and the density function calculated in three dimensional space, which is t -constant of motion, and also interpret the meaning of this relation.

2 Special Relativistic Quantum Mechanics

The *ensemble's* state is described by the functions $F(x^\alpha, p^\alpha)$ where x^α and p^α are the position and momentum four-vectors of each particle belonging to a system of the *ensemble*.

Let us list the modified axioms of our theory

(A1') Special relativistic mechanics of particles is valid for all particles of the *ensemble's* component systems.

(A2') For isolated system *ensembles*, the joint probability density function is a τ -constant of motion

$$\frac{d}{d\tau} F(x^\alpha, p^\alpha) = 0, \quad (1)$$

where τ is the proper time.

(A3') The Wigner-Moyal Infinitesimal Transformation, defined as

$$\begin{aligned} \rho \left(x^\alpha + \frac{\delta x^\alpha}{2}, x^\alpha - \frac{\delta x^\alpha}{2} \right) &= \int F(x^\alpha, p^\alpha) \exp \left(i \frac{p^\beta \delta x_\beta}{2} \right) \cdot \\ &\cdot \exp \left[\frac{ie}{\hbar c} \int_0^{x+\frac{\delta x}{2}} A^\lambda(u) du_\lambda + \frac{ie}{\hbar c} \int_0^{x-\frac{\delta x}{2}} A^\lambda(u) du_\lambda \right] d^4 p, \end{aligned} \quad (2)$$

where we include, for generality, an electromagnetic field through the four-vector

$$A^\lambda = (\phi, \mathbf{A}), \quad (3)$$

where ϕ is the scalar potential and \mathbf{A} the vector potential, is adequate for the description of a general quantum system in the presence of electromagnetic fields.

With equation (1), we can write

$$\frac{dx^\alpha}{d\tau} \frac{\partial F}{\partial x^\alpha} + \frac{dp^\alpha}{d\tau} \frac{\partial F}{\partial p^\alpha} = 0. \quad (4)$$

We can also use axiom (A1') to write the particle's relativistic equations

$$\frac{dx^\alpha}{d\tau} = \frac{p^\alpha}{m} \quad ; \quad \frac{dp^\alpha}{d\tau} = f^\alpha = -\frac{\partial V}{\partial x_\alpha}. \quad (5)$$

Using the transformation (2) in (4), we reach the expression

$$\begin{aligned} \frac{1}{2m} \left\{ \left[i\hbar \frac{\partial}{\partial y^\alpha} + \frac{e}{c} A_\alpha(y) \right]^2 - \left[i\hbar \frac{\partial}{\partial y'^\alpha} + \frac{e}{c} A_\alpha(y') \right]^2 \right\} \rho - \\ - [V(y) - V(y')] \rho = 0, \end{aligned} \quad (6)$$

where we once again use

$$\frac{\partial V}{\partial x_\alpha} \delta x_\alpha = V \left(x + \frac{\delta x}{2} \right) - V \left(x - \frac{\delta x}{2} \right), \quad (7)$$

along with the following change of variables

$$y^\alpha = x^\alpha + \frac{\delta x^\alpha}{2} \quad ; \quad y'^\alpha = x^\alpha - \frac{\delta x^\alpha}{2}. \quad (8)$$

If we ignore the potential term, equation (6) is the Klein-Gordon's density function equation for a spinless particle in the presence of an electromagnetic field. If we are dealing with particles capable of coupling to external electric and magnetic fields through their electric and magnetic moments, $\vec{\pi}$ and $\vec{\mu}$ respectively, then the interaction force which is a Lorentz scalar, is given, in a first approximation, by

$$F_{int}^\alpha = -\partial^\alpha (\vec{\pi} \cdot \mathbf{E} + \vec{\mu} \cdot \mathbf{B}). \quad (9)$$

Equation (6) becomes

$$\begin{aligned} & \frac{1}{2m} \left\{ \left[i\hbar \frac{\partial}{\partial y^\alpha} + \frac{e}{c} A_\alpha(y) \right]^2 - \left[i\hbar \frac{\partial}{\partial y'^\alpha} + \frac{e}{c} A_\alpha(y') \right]^2 \right\} \rho - \\ & - [(\vec{\pi} \cdot \mathbf{E} + \vec{\mu} \cdot \mathbf{B})(y) - (\vec{\pi} \cdot \mathbf{E} + \vec{\mu} \cdot \mathbf{B})(y')] \rho = 0, \end{aligned} \quad (10)$$

which we call Dirac's First Equation for the density function.

The imposition that the potential in (10) be a Lorentz scalar is enough for us to construct a tensor associated to the internal degrees of freedom - internal moments. Landé's factor, cited in the last paper, can be obtained, as usual, passing to the non-relativistic limit[24] of equation (10) above.

In order to obtain an equation for the probability amplitude we can write, in a way similar to that done in the first paper (hereafter identified as (I)),

$$\rho(y^\alpha, y'^\alpha) = \Psi^*(y'^\alpha) \Psi(y^\alpha), \quad (11)$$

where

$$\Psi(y^\alpha) = R(y^\alpha) \exp \left[\frac{i}{\hbar} S(y^\alpha) \right], \quad (12)$$

being $R(y)$ and $S(y)$ real functions. Using the change in variables (8) and expanding expression (11) up to the second order in δx , we obtain

$$\begin{aligned} & \rho \left(x^\alpha + \frac{\delta x^\alpha}{2}, x^\alpha - \frac{\delta x^\alpha}{2} \right) = \exp \left[\frac{i}{\hbar} \frac{\partial S}{\partial x^\beta} \delta x^\beta \right] \cdot \\ & \cdot \left\{ R(x^\alpha)^2 - \left(\frac{\delta x^\beta}{2} \right)^2 \left[\left(\frac{\partial R}{\partial x^\alpha} \right)^2 - R \frac{\partial^2 R}{\partial x_\beta \partial x^\beta} \right] \right\}. \end{aligned} \quad (13)$$

Substituting this expression in equation (6), written in terms of x and δx and without including, for the sake of simplicity, the electromagnetic potentials, we get

$$-\frac{\hbar^2}{m} \frac{\partial^2 \rho}{\partial x^\alpha \partial (\delta x_\alpha)} - \frac{\partial V}{\partial x^\alpha} \delta x^\alpha \rho = 0 \quad (14)$$

and, holding the zero and first order terms in δx , we reach the equation

$$\frac{i}{\hbar} \partial_\alpha \left(R^2 \frac{\partial^\alpha S}{m} \right) + \delta x^\alpha \partial_\alpha \left\{ \frac{-\hbar^2}{2mR} \square R + V + \frac{\partial_\beta S \partial^\beta S}{2m} \right\} = 0, \quad (15)$$

where we use $\partial_\alpha = \partial/\partial x^\alpha$ and $\square = \partial_\alpha \partial^\alpha$. Collecting the real and complex terms and equating them to zero, we get the pair of equations

$$\partial_\alpha \left(R^2 \frac{\partial^\alpha S}{m} \right) = 0, \quad (16)$$

$$\frac{-\hbar^2}{2mR} \square R + V + \frac{\partial_\beta S \partial^\beta S}{2m} = \text{const.} \quad (17)$$

The constant in (17) can be obtained using a relativistic solution for the free particle. In this case, it is easy to demonstrate that the constant will be given by

$$\text{const.} = \frac{mc^2}{2}, \quad (18)$$

so that equation (17) becomes [25, 26]

$$\frac{-\hbar^2}{2mR} \square R + V - \frac{mc^2}{2} + \frac{\partial_\beta S \partial^\beta S}{2m} = 0. \quad (19)$$

Reintroducing the electromagnetic potentials, this equation is formally identical to the equation

$$\left\{ \frac{1}{2m} \left[i\hbar \frac{\partial}{\partial x^\alpha} + \frac{e}{c} A_\alpha(x) \right]^2 + V(x) + (\vec{\pi} \cdot \mathbf{E} + \vec{\mu} \cdot \mathbf{B})(x) - \frac{mc^2}{2} \right\} \Psi(x) = 0, \quad (20)$$

since the substitution of expression (12) in the equation above gives us equation (19), when the electromagnetic potentials are considered. We call this equation, without the potential term, Klein-Gordon's Second Equation, while for a potential such as in (9), we call it Dirac's Second Equation for the probability amplitude.

In order to obtain mean values in relativistic phase space of some function $\Theta(x, p)$, we should calculate the integral

$$\overline{\Theta(x^\alpha, p^\alpha)} = \lim_{\delta x \rightarrow 0} \int O_p(x^\alpha, \delta x^\alpha) \rho \left(x^\alpha + \frac{\delta x^\alpha}{2}, x^\alpha - \frac{\delta x^\alpha}{2} \right) d^4 x. \quad (21)$$

Following the same steps as in (I), we can introduce the four-momentum and four-position operators as being

$$\hat{p}'_\alpha = -i\hbar \frac{\partial}{\partial(\delta x^\alpha)} \quad ; \quad \hat{x}'_\alpha = x_\alpha. \quad (22)$$

Note that we are calculating integrals in relativistic four-spaces, for it is in these spaces that the density function is τ -conserved. If we desire to calculate values in habitual three dimensional configuration space through the probability amplitudes we may, as is habitual, take equation (20) for ψ , multiply it to the left by ψ^* and subtract it from the equation for ψ^* , multiplied for the left by ψ , to obtain (in the absence of electromagnetic fields)

$$j^\alpha = \frac{i\hbar}{2m} [\psi^* \partial^\alpha \psi - \psi \partial^\alpha \psi^*], \quad (23)$$

which we define as the four-current. In this case, we have the continuity equation

$$\partial_\alpha j^\alpha = 0, \quad (24)$$

formally equivalent to equation (16) if we use the decomposition (12) for the amplitudes. In the appendix we present a different technique to obtain this current through the density function which provides us with the correct interpretation of $j^\alpha(x)$. We can then take the zero component of the four-current

$$P(x) = \frac{i\hbar}{2m} [\psi^* \partial^0 \psi - \psi \partial^0 \psi^*] \quad (25)$$

as being the probability density in three dimensional space, since it reduces itself to the correct non-relativistic probability density in the appropriate limit[27]. The results referring to the existence of particles and anti-particles are the usual and will be shortly discussed in the appendix together with the positivity of equation (25).

From equation (16) and (17), we can calculate a statistical potential, analogous to the one obtained in the previous article. In this case it is easy to show that this potential is given by

$$V_{eff}(x) = V(x) - \frac{\hbar^2}{2mR} \square R, \quad (26)$$

and is associated to the equation

$$\frac{dp^\alpha}{d\tau} = -\partial^\alpha V_{eff}(x), \quad (27)$$

together with the initial condition

$$p^\alpha = \partial^\alpha S. \quad (28)$$

These last three expressions will be very useful in the next section where we will undertake the general relativistic treatment.

3 General Relativistic Quantum Mechanics

The axioms should again be altered in order to make them adequate for the general theory of relativity. Let us list our axioms bellow:

(A1'') The general relativistic mechanics of particles is valid for all particles of the *ensemble's* component systems.

(A2'') For an *ensemble* of single particle isolated systems in the presence of a gravitational field, the joint probability density function representing this *ensemble* is a conserved quantity when its variation is taken along the system's geodesics, that is

$$\frac{DF(x^\alpha, p^\alpha)}{D\tau} = 0, \quad (29)$$

where τ is the proper time associated to the geodesic and $D/D\tau$ is the derivative taken along the geodesic defined by τ .

(A3'') The Wigner-Moyal Infinitesimal Transformation defined as

$$\rho\left(x^\alpha + \frac{\delta x^\alpha}{2}, x^\alpha - \frac{\delta x^\alpha}{2}\right) = \int F(x^\alpha, p^\alpha) \exp\left(\frac{i}{\hbar} p^\beta \delta x_\beta\right) d^4 p \quad (30)$$

is valid for the description of any quantum system in the presence of gravitational fields.

With equation (29), we can write

$$\frac{Dx^\alpha}{D\tau} \nabla_{x^\alpha} F + \frac{Dp^\alpha}{D\tau} \nabla_{p^\alpha} F = 0, \quad (31)$$

and using axiom (A1''), we have

$$\frac{Dx^\alpha}{D\tau} = \frac{p^\alpha}{m} \quad ; \quad \frac{Dp^\alpha}{D\tau} = f^\alpha. \quad (32)$$

Using now the transformation (30) and the usual change of variables

$$y^\alpha = x^\alpha + \frac{\delta x^\alpha}{2} \quad ; \quad y'^\alpha = x^\alpha - \frac{\delta x^\alpha}{2}, \quad (33)$$

we reach the generalized relativistic quantum equation for the density function

$$\left\{ \frac{-\hbar^2}{2m} [\nabla_\alpha^2 - \nabla_{\alpha'}^2] + [V(y) - V(y')] \right\} \rho = 0, \quad (34)$$

where ∇_α and $\nabla'_{\alpha'}$ are the covariant derivatives according to y and y' respectively.

Assuming the validity of the decomposition

$$\rho(y', y) = \Psi^*(y') \Psi(y), \quad (35)$$

with

$$\Psi(y) = R(y) \exp[iS(y)/\hbar], \quad (36)$$

we obtain the following pair of expressions

$$\nabla_\mu \left[R(x)^2 \frac{\nabla^\mu S}{m} \right] = 0, \quad (37)$$

$$\frac{-\hbar^2}{2mR} \square R + V - \frac{mc^2}{2} + \frac{\nabla_\beta S \nabla^\beta S}{2m} = 0, \quad (38)$$

where now $\square = \nabla^\mu \nabla_\mu$.

Obtaining, as in the previous section, the expression for the potential and the probability force associated with the "statistical field"

$$V_{(Q)}(x) = \frac{-\hbar^2}{2mR} \square R \quad ; \quad f_{(Q)}^\mu = \nabla^\mu V_{(Q)}(x) \quad (39)$$

we can write

$$m \frac{D^2 x^\mu}{D\tau^2} = f^\mu(x) + f_{(Q)}^\mu(x) \quad (40)$$

which is an equation for the *ensemble*. In this case, equation (40) can be considered the equation for the possible geodesics associated to the different configurations which the *ensemble's* systems may possess. Nevertheless, we must stress that each system's particle still obeys equation (32) strictly.

According to Einstein's intuition, we put

$$G_{\mu\nu} = -\frac{8\pi G}{c^2} [T_{(M)\mu\nu} + T_{(Q)\mu\nu}], \quad (41)$$

where $G_{\mu\nu}$ is Einstein's tensor, $T_{(M)\mu\nu}$ is the energy-momentum tensor associated to the forces represented by $f^\mu(x)$ in equation (40) and $T_{(Q)\mu\nu}$ is the tensor associated to the statistical potential.

The tensor $T_{(Q)\mu\nu}$ can be obtained looking at equations (37) and (38). Equation (38) represents the possible geodesics related to the *ensemble*, as was pointed out above, and equation (37) defines an equation for the "statistical" field variables $R(x)$ and $S(x)$. The tensor associated with this equation is given by

$$T_{(Q)\mu\nu} = mR(x)^2 \left[\frac{\nabla_\mu S}{m} \frac{\nabla_\nu S}{m} \right] \quad (42)$$

and is called a matter tensor if we make the following substitution

$$u_\mu = \frac{\nabla_\mu S}{m} \quad ; \quad \rho'(x) = mR(x)^2, \quad (43)$$

as some kind of statistical four-velocity and statistical matter distribution respectively, to get[28]

$$T_{(Q)\mu\nu} = \rho'(x) u_\mu u_\nu \quad (44)$$

The interpretation of this tensor is quite simple and natural. It represents the statistical distribution of matter in space-time.

The system of equations to be solved is

$$\frac{-\hbar^2}{2mR} \square R + V - \frac{mc^2}{2} + \frac{\nabla_\beta S \nabla^\beta S}{2m} = 0, \quad (45)$$

$$G_{\mu\nu} = -\frac{8\pi G}{c^2} [T_{(M)\mu\nu} + T_{(Q)\mu\nu}], \quad (46)$$

This system must be solved in the following way. First we solve Einstein's equation for the, yet unknown, probability density $\rho'(x)$ and obtain the metric in terms of this function. With the metric at hand, expressed in terms of the functions $R(x)$ and $S(x)$, we return to equation (45) and solve it for these functions. The Schwarzschild general relativistic quantum mechanical problem was already solved using this system of equations and the results will be published elsewhere.

One important thing to note is that system (45,46) is highly non-linear and in general will not present quantization or superposition effects. Also, when the quantum mechanical system is solved, we will have a metric at hand that reflects the probabilistic character of the calculations. This metric *does not* represent the real metric of space-time; it represents a statistical behavior of space-time geometry as related to the initial conditions imposed on the component systems of the considered *ensemble*.

We will return to this matters in the last paper when epistemological considerations will take place.

4 Conclusion

In this paper we derived all of special relativistic quantum mechanics for single particle systems. Once again, we note that it is Dirac's second order equation that is obtained; beyond this, there is nothing in the formalism which allows us to obtain the first order equation from the second order one through a projection operation, as is usually done. We also note that the solutions of the linear equation are also solutions of the second order equation, but the inverse is not true. We are thus forced by the formalism to view Dirac's second order equation as the fundamental one, contrary to what is accepted in the literature. It is interesting to note that there was never any reason, other than historical, to accept Dirac's linear equation as fundamental for relativistic quantum mechanics. This means also that we do not need to accept as real the interpretation

of the vacuum as an antiparticle sea, since there is no need for such an entity when second order equations are considered.

It was also possible to obtain a quantum mechanical general relativistic equation which takes the action of gravitational fields into account. The equations obtained pointed in the direction of quantization extinction by strong gravitational fields.

The next paper will discuss the epistemological implications of these results.

A Three-Dimensional Probability Densities

We have seen that we can define momentum-energy and space-time operators as

$$\hat{p}'_{\alpha} = -i\hbar \frac{\partial}{\partial(\delta x^{\alpha})} \quad ; \quad \hat{x}'_{\alpha} = x_{\alpha}, \quad (47)$$

acting upon the density function. For these operators any function's mean values are calculated using integrals defined on the volume element d^4x . In relativistic quantum mechanic's usual treatment, the probability density is defined by the expression

$$P(x) = j^0(x) = \frac{i\hbar}{2m} [\psi^* \partial^0 \psi - \psi \partial^0 \psi^*], \quad (48)$$

as already defined in (25). We now want to interpret this result and find a connection between the density function $\rho(y', y)$, which is τ -conserved, and the zero component of the four-current $P(x)$, which is t -conserved.

To do this we start noting that the expression for the mean four-momentum is given by

$$\overline{p^{\alpha}} = \lim_{\delta x \rightarrow 0} \int -i\hbar \frac{\partial}{\partial(\delta x_{\alpha})} \rho \left(x^{\alpha} + \frac{\delta x^{\alpha}}{2}, x^{\alpha} - \frac{\delta x^{\alpha}}{2} \right) d^4x. \quad (49)$$

Supposing that we can decompose the density function according to

$$\rho \left(x^{\alpha} + \frac{\delta x^{\alpha}}{2}, x^{\alpha} - \frac{\delta x^{\alpha}}{2} \right) = \Psi^* \left(x^{\alpha} + \frac{\delta x^{\alpha}}{2} \right) \Psi \left(x^{\alpha} - \frac{\delta x^{\alpha}}{2} \right) \quad (50)$$

and substituting it in expression (49), it can be shown that

$$\overline{p^{\alpha}} = \int \frac{\hbar}{2i} [\Psi(x) \partial^{\alpha} \Psi^*(x) - \Psi^*(x) \partial^{\alpha} \Psi(x)] d^4x, \quad (51)$$

or, in terms of the four-current

$$\overline{p^{\alpha}} = \int mcj^{\alpha}(x) d^4x \quad (52)$$

Thus, for a closed system, we can guarantee that the integral of the zero component of the above vector, the energy, will not vary in time. In fact, since

$$\partial_\alpha j^\alpha = 0, \quad (53)$$

we can write the above integral as an integral only in three dimensional space[29]. To obtain a dimensionless value, we can divide the expression (52) by $\pm mc$. In this manner, we guarantee that the integral of the zero component of the four-vector in (52) is a t -conserved dimensionless function. And more, it can be shown that the term $j^0(x)$ reduces to the probability density in the non-relativistic limit.

Nevertheless, there is one last step towards the acceptance of $j^0(x)$ as a probability density. This function, as is well known, can have both positive and negative values. This is expected if we consider that relativistic energy has this characteristic. That's why we dived by $\pm mc$ for the particle and anti-particle respectively to obtain positive definite probabilities. With this procedure we distinguish particles and anti-particles in the mathematical formalism without any need to multiply by electronic charges as is usually done.

With these conventions we have obtained the probability density for three dimensional space.

References

- [1] Moyal, J. E., Proc. Camb. Phyl. Soc. **45**, 99 (1949)
- [2] Husimi, K., Proc. Phys. Math. Soc. Japan **22**, 264 (1940)
- [3] Groenewold, H. J., Physica **12**, 405 (1946)
- [4] Weyl, H., " *The Theory of Groups and Quantum Mechanics*" (Dover Publ. Co., N.Y., pp. 274-275, 1950)
- [5] Bopp, F., Ann. Inst. Henri Poincaré **15**, 81 (1956)
- [6] Margenau, H. and Hill, R. N., " *Progress in Theoretical Physics*" (Kyoto) **26**, 722 (1961)
- [7] Segal, I., Can. J. Math. **13**, 1 (1961)
- [8] Metha, C. L., J. Math. Phys. **5**, 677 (1964)
- [9] Metha, C. L., Phys. Rev. **138**, B274 (1965)
- [10] Pool, J. C. T., J. Math. Phys **7**, 66 (1966)
- [11] Misra, S. P. and Shankara, T. S., J. Math. Phys. **9**, 299 (1968)
- [12] Wigner, E. P., Phys. Rev. **40**, 749 (1932)

- [13] Hillery, M., O'Connell, R. F., Scully, M. O. and Wigner, E. P., Phys. Reports **106**, 121-167 (1984)
- [14] Cohen, L., Philosophy of Science **33**, 317 (1966)
- [15] Margenau, H. and Cohen, L., in "*Quantum Theory and Reality*", eds. W. Yourgrau and A. van der Merwe (MIT press, Cambridge, pp. 25-36, 1971)
- [16] Ali, S. T. and Prugovecki, E., J. Math. Phys. **18**, 219 (1977)
- [17] Ali, S. T. and Prugovecki, E., Physica **89A**, 501 (1977)
- [18] Ali, S. T. and Prugovecki, E., Intern. J. Theor. Phys. **16**, 689 (1977)
- [19] Prugovecki, E., Annals of Phys. **110**, 102 (1978)
- [20] Prugovecki, E., Physica **91A**, 202 (1978)
- [21] Prugovecki, E., Phys. Rev. **D18**, 3655 (1978)
- [22] Prugovecki, E., Found. of Phys. **9**, 575 (1979)
- [23] Prugovecki, E., Hadronic Journal **4**, 1118 (1981)
- [24] Dewdney, C., Holland, P.R., Kyprianidis, A., Maric, Z. and Vigier, J.P., Phys. Lett. **113A**, 359 (1986)
- [25] Baym, G., "*Lectures on Quantum Mechanics*" (The Benjamin/Cummings Publishing Co., CA, 1973)
- [26] Kyprianidis, A., Phys. Lett. **111A**, 111 (1985)
- [27] Schiff, L., "*Quantum Mechanics*" (McGraw-Hill Book Co., Singapore, Philippines, 1968)
- [28] Weinberg, S. "*Gravitation and Cosmology, principles and applications of the general theory of relativity*" (John Willey and Sons, Inc., N.Y., 1972)
- [29] Goldstein, H., "*Classical Mechanics*" (Addison Wesley Publishing Co., Philippines, 1990)